

Cohomology Operators Defined by a Deformation

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INTRODUCTION

A lot of (co)homological information on modules over a commutative ring R is encoded in terms of composition products of various Ext and Tor modules. Two main difficulties in using this information are that the resulting algebra and module structures are seldom finite, and the products are almost never commutative. One significant exception occurs when $R = Q/(\mathbf{x})$ for a Koszul-regular set $\mathbf{x} = \{x_1, \dots, x_c\}$ in a commutative ring Q ; we think of Q as a deformation of R over a regular base.

Indeed, Gulliksen [8] then constructs a set of commuting operators $\{X_1, \dots, X_c\}$ acting on $\text{Ext}_R^*(M, N)$ by *increasing* degrees by 2 and on $\text{Tor}_R^*(M, N)$ by *decreasing* degrees by 2, making Ext and Tor graded modules over a polynomial ring $\mathcal{S} = R[X_1, \dots, X_c]$ with variables of cohomological degree 2. He proves that if $\text{Ext}_Q^n(M, N)$ is noetherian over Q for each n and vanishes for $n \gg 0$, then the graded \mathcal{S} -module $\text{Ext}_R^*(M, N)$ is noetherian. This partly overcomes the first obstacle referred to above.

Under more restrictive conditions on Q and \mathbf{x} , Mehta [9] interprets Gulliksen's operators as composition products: this makes the second difficulty manageable, as the actions of $\text{Ext}_R^*(M, M)$ and $\text{Ext}_R^*(N, N)$ on $\text{Ext}_R^*(M, N)$ are “essentially central.”

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These results have provided the basis for extensive studies of the (co)homological properties of modules over local complete intersections by Eisenbud [7], and of more general classes of modules of infinite projective dimension by Avramov [1].

Each of the papers quoted above also uses alternative construction(s) of cohomology operators. It has been repeatedly stated in the literature that they yield the same result, but a close inspection of the arguments supporting such claims has revealed serious defects. On the other hand, no single approach seems to give all the essential properties of the cohomology operators which have been extensively used in [4].

In an attempt to provide complete proofs for the coincidence of the various operators, we were led to two new constructions. Both of them come from viewing R -modules as DG (= differential graded) modules over the Koszul complex K resolving R over Q , and our arguments make a full-fledged use of techniques of DG homological algebra, summarized in Section 1. As a bonus, we construct the operators and establish their main properties directly for *complexes* of R -modules.

We first introduce operators *à la* Gulliksen, as connecting homomorphisms. Unlike previous approaches, *both* module arguments are involved from the start. This is important in proving that “left” and “right” versions of earlier operators agree—at least up to sign—which suffices for the applications in [7, 1].

On our second approach we produce cohomology operators from chain-level maps, as did Eisenbud. However, our chain endomorphisms arise not from lifting the differential of a complex of R -modules, but from descending the differential of a DG K -module.

These constructions are presented in Section 2. They are used in Section 3 to provide direct proofs of many formal properties of the cohomology operators, in particular, of their *centrality*. Section 4 compares our approach to earlier ones, filling in gaps and correcting misconceptions concerning the relations between the various operators. Trying to avoid further inaccuracies, we include sufficient detail for most of the essential computations.

In Section 5, we note that some of the existing proofs of the *noetherian nature* of $\text{Ext}_R^*(M, N)$ over the ring of cohomology operators carry over to complexes, and establish a new property of these operators—their *primitivity*. When k is the residue field of R , for certain finite R -modules M we prove that $\text{Ext}_R^*(M, k)$ is a finite graded module over the subalgebra of $\text{Ext}_R^*(k, k)$ generated by the central and primitive elements of degree 2, even though R itself may not have a deformation. This result plays a key role in the investigation of finite CI-dimension in the recent paper [4].

1. BACKGROUND

This section contains a synopsis of DG homological algebra, taken from [3]. All modules in sight are defined over a fixed commutative ring \mathbb{k} , which is usually suppressed from the notation. In particular, \otimes stands for $\otimes_{\mathbb{k}}$ and Hom for $\text{Hom}_{\mathbb{k}}$.

1.1. Complexes and DG Modules. Differentials have degree -1 and are ubiquitously denoted ∂ . The notation $m \in M$ means that m is a homogeneous element of M ; we denote its degree by $|m|$. If m is a cycle, then $[m]$ is its homology class. The functor forgetful of differentials is denoted $(-)^{\natural}$.

Let M and N be complexes. The complex $\text{Hom}(M, N)$ has d th component $\prod_{i \in \mathbb{Z}} \text{Hom}(M_i, N_{i+d})$, and differential $\partial(\gamma) = \partial \circ \gamma - (-1)^{|\gamma|} \gamma \circ \partial$. A chain map γ is a cycle in $\text{Hom}(M, N)$, so that $\partial \circ \gamma = (-1)^{|\gamma|} \gamma \circ \partial$. A chain map of degree 0 is called a *morphism*, and a *quasi-isomorphism* if it induces an isomorphism in homology; the latter is indicated by the appearance of the symbol of \simeq next to its arrow. The complex $M \otimes N$ has d th component $\prod_i (M_i \otimes N_{d-i})$ and differential $\partial(m \otimes n) = \partial(m) \otimes n + (-1)^{|m|} m \otimes \partial(n)$.

A commutative DG algebra A is a complex together with a morphism $A \otimes A \rightarrow A$ denoted by juxtaposition which has the usual associativity and unit properties, and satisfies $ab = (-1)^{|a||b|}ba$ for all $a, b \in A$ and $a^2 = 0$ when $|a|$ is odd. An A -module is a complex M , together with a morphism $A \otimes M \rightarrow M$, satisfying the usual conditions. If $A_i = 0$ for $i \neq 0$ then a DG A -module is simply a complex of modules over the ring A_0 , and we call it accordingly.

For A -modules M and N , the subcomplex $\text{Hom}_A(M, N)$ of $\text{Hom}(M, N)$ consists of those γ for which $\gamma(am) = (-1)^{|\gamma||a|}a\gamma(m)$ for all $a \in A$ and $m \in M$; it is an A -module with $(a\gamma)(m) = a(\gamma(m)) = (-1)^{|\gamma||a|}\gamma(am)$. Dually, $M \otimes_A N$ is the quotient of $M \otimes N$ by the subcomplex generated by $\{(am) \otimes n - (-1)^{|a||m|}m \otimes (an) \mid a \in A, m \in M, n \in N\}$; it is an A -module, with $a(m \otimes n) = (am) \otimes n = (-1)^{|a||m|}m \otimes (an)$.

1.2. Resolutions. An A -module P is *DG-projective* if $\text{Hom}_A(P, -)$ transforms surjective quasi-isomorphisms into surjective quasi-isomorphisms. This is equivalent to the property that $\text{Hom}_A(P, -)$ preserves all quasi-isomorphisms and the graded A^{\natural} -module P^{\natural} is projective. In that case $(P \otimes_A -)$ preserves quasi-isomorphisms.

An A -module I is *DG-injective* if $\text{Hom}_A(-, I)$ transforms injective quasi-isomorphisms into surjective quasi-isomorphisms. This is equivalent to the properties that $\text{Hom}_A(-, I)$ preserves quasi-isomorphisms and the graded A^{\natural} -module I^{\natural} is injective.

A bounded below (respectively, above) A -module M such that M^\natural is projective (respectively, injective) over A^\natural is DG-projective (respectively, DG-injective).

Let X be an arbitrary A -module. There exists a quasi-isomorphism $P \xrightarrow{\sim} X$ from a DG-projective module P ; any such morphism is called a *DG-projective resolution* of X . Dually, X has a *DG-injective resolution*, that is, a quasi-isomorphism $X \xrightarrow{\sim} I$ with I DG-injective. For each X we fix a *DG-projective resolution* $\varepsilon_X^A: P_X^A \xrightarrow{\sim} X$ and a *DG-injective resolution* $\eta_A^X: X \xrightarrow{\sim} I_A^X$.

Let $\varphi: A' \rightarrow A$ be a homomorphism of DG algebras, let M and N be A -modules, and let M' and N' be A' -modules. A φ -contravariant homomorphism $\nu: N \rightarrow N'$ is a \mathbb{K} -linear map such that $\nu(\varphi(a')n) = (-1)^{|\nu||a'|}a'\nu(n)$ for all $a' \in A'$ and all $n \in N$. A φ -covariant homomorphism $\mu: M' \rightarrow M$ is a \mathbb{K} -linear map such that $\mu(a'm') = (-1)^{|\mu||a'|}\varphi(a')\mu(m')$ for all $a' \in A'$ and all $m' \in M'$. In other words, ν and μ are homomorphisms of A' -modules for the structures induced on N and M through φ .

For φ, μ, ν , as above, and a φ -covariant chain map $\lambda: L' \rightarrow L$, by Subsection 1.2 there exist unique up to homotopy φ -equivariant chain maps λ^*, μ^*, ν^* , such that the squares

$$\begin{array}{ccccc} P_{L'}^{A'} & \xrightarrow{\lambda^*} & P_L^A & & P_{M'}^{A'} & \xrightarrow{\mu^*} & P_M^A & & N & \xrightarrow{\nu^*} & N' \\ \simeq \downarrow & & \downarrow \simeq & & \simeq \downarrow & & \downarrow \simeq & & \simeq \downarrow & & \downarrow \simeq \\ L' & \xrightarrow{\lambda} & L & & M' & \xrightarrow{\mu} & M & & I_A^N & \xrightarrow{\nu} & I_{A'}^{N'} \end{array}$$

commute up to homotopy. They induce a φ -contravariant chain map

$$\mathrm{Hom}_\varphi(\mu^*, \nu^*): \mathrm{Hom}_A(P_M^A, I_A^N) \rightarrow \mathrm{Hom}_{A'}(P_{M'}^{A'}, I_{A'}^{N'}),$$

$$\gamma \mapsto (-1)^{|\mu|(|\gamma|+|\nu|)} \nu^* \gamma \mu^*.$$

and a φ -covariant chain map

$$\lambda^* \otimes_\varphi \mu^*: P_{L'}^{A'} \otimes_{A'} P_{M'}^{A'} \rightarrow P_L^A \otimes_A P_M^A,$$

$$p' \otimes q' \mapsto (-1)^{|\mu||p'|} \lambda^*(p') \otimes \mu^*(q')$$

both of which are unique up to homotopy.

1.3. Derived Functors. With the resolutions chosen in Subsection 1.2, set $\mathrm{Ext}_A^*(M, N) = \mathrm{H}\mathrm{Hom}_A(P_M^A, I_A^N)$ and $\mathrm{Tor}_*^A(L, M) = \mathrm{H}(P_L^A \otimes_A P_M^A)$. The morphisms from Subsection 1.2 uniquely define homomorphisms of

graded \mathbb{K} -modules

$$\mathrm{Ext}_{\varphi}^*(\mu, \nu): \mathrm{Ext}_A^*(M, N) \rightarrow \mathrm{Ext}_{A'}^*(M', N')$$

and

$$\mathrm{Tor}_{*}^{\varphi}(\lambda, \mu): \mathrm{Tor}_{*}^{A'}(L', M') \rightarrow \mathrm{Tor}_{*}^A(L, M)$$

which make Ext^* and Tor_* functors of three arguments. If $\varphi, \lambda, \mu, \nu$, are quasi-isomorphisms, then $\mathrm{Ext}_{\varphi}^*(\mu, \nu)$ and $\mathrm{Tor}_{*}^{\varphi}(\lambda, \mu)$ are isomorphisms.

When $A = A_0$ is a ring, the construction yields the usual hyper (co)homology of complexes. By further specialization to modules $M = M_0$ and $N = N_0$, it produces the classical derived functors of Hom_A and \otimes_A .

Let $\gamma \in \mathrm{Ext}_A^*(M, N)$ and $\delta \in \mathrm{Ext}_A^*(L, M)$ be represented by chain maps $\tilde{\gamma}: P_M^A \rightarrow I_A^N$ and $\tilde{\delta}: P_L^A \rightarrow I_A^M$. By Subsection 1.2, choose unique up to homotopy chain maps $\gamma: P_M^A \rightarrow P_N^A$ and $\delta: P_L^A \rightarrow P_M^A$ such that $\tilde{\gamma} = \eta_A^N \circ \varepsilon_N^A \circ \gamma$ and $\tilde{\delta} = \eta_A^M \circ \varepsilon_M^A \circ \delta$.

The composition product is the degree 0 homomorphism of graded \mathbb{K} -modules

$$\mathrm{Ext}_A^*(M, N) \otimes \mathrm{Ext}_A^*(L, M) \rightarrow \mathrm{Ext}_A^*(L, N),$$

$$\gamma \otimes \delta \mapsto \gamma \cdot \delta = [\tilde{\gamma} \circ \tilde{\delta}].$$

If t is the homology class of a cycle $\sum_k e_k \otimes_A f_k \in P_L^A \otimes_A P_M^A$, then

$$\mathrm{Ext}_A^*(M, N) \otimes \mathrm{Tor}_*^A(L, M) \rightarrow \mathrm{Tor}_*^A(L, N),$$

$$\gamma \otimes t \mapsto \gamma \cdot t = \left[\sum_k (-1)^{|\gamma||e_k|} e_k \otimes_A \gamma(f_k) \right]$$

defines a degree 0 homomorphism of graded \mathbb{K} -modules.

Similarly, if u is the homology class of $\sum_k e_k \otimes_A g_k \in P_L^A \otimes_A P_N^A$, then

$$\mathrm{Ext}_A^*(L, M) \otimes \mathrm{Tor}_*^A(L, N) \rightarrow \mathrm{Tor}_*^A(M, N),$$

$$\delta \otimes u \mapsto \delta \cdot u = \left[\sum_k \delta(e_k) \otimes_A g_k \right]$$

is a degree 0 homomorphism of graded \mathbb{K} -modules.

These pairings are associative, and so define a structure of graded \mathbb{K} -algebra on $\mathrm{Ext}_A^*(M, M)$ with the unit given by the class of the map $\eta_A^M \circ \varepsilon_M^A: P_M^A \rightarrow I_A^M$; they make $\mathrm{Ext}_A^*(M, N)$ into a graded left $\mathrm{Ext}_A^*(N, N)$ -, right $\mathrm{Ext}_A^*(M, M)$ -bimodule, while $\mathrm{Tor}_*^A(L, M)$ becomes a graded left $\mathrm{Ext}_A^*(L, L)$ -, left $\mathrm{Ext}_A^*(M, M)$ -bimodule.

2. CONSTRUCTION

We start by describing the setup for much of the discussion in this paper.

2.1. Koszul Regularity. Let $\mathbf{x} = \{x_1, \dots, x_c\}$ be a set of elements in Q , $\alpha = (\mathbf{x})$ the ideal it generates, and $R = Q/\alpha$ the quotient ring. We denote by K the Koszul complex $K(\mathbf{x}, Q)$; thus, K^\natural is the exterior algebra on a free Q -module with basis ξ_1, \dots, ξ_c with $|\xi_j| = 1$, and $\partial(\xi_j) = x_j$ for $1 \leq j \leq c$. We denote \mathfrak{U} the kernel of the canonical augmentation $\kappa: K \rightarrow H_0(K) = R$.

We assume that \mathbf{x} is Koszul-regular, that is, that $H_i(K) = 0$ for $i \neq 0$; equivalently, κ is a quasi-isomorphism. The cotangent module $C = \alpha/\alpha^2$ is then free over R on $\bar{\mathbf{x}}$, where $\bar{x}_j = x_j + \alpha^2$, and for each R -module M and each $i \in \mathbb{Z}$ there are isomorphisms

$$\begin{aligned} \mathrm{Ext}_Q^i(R, M) &\simeq H_{-i}\mathrm{Hom}_Q(K, M) = \mathrm{Hom}_Q(K_i, M) \\ &\cong \bigwedge^i \mathrm{Hom}_R(C, R) \otimes_R M \end{aligned}$$

and

$$\mathrm{Tor}_i^Q(R, M) \simeq H_i(K \otimes_Q M) = K_i \otimes_Q M \cong \bigwedge^i C \otimes_R M.$$

The canonical isomorphisms $\kappa^* = \mathrm{Ext}_\kappa^*(M, N)$ and $\kappa_* = \mathrm{Tor}_*^\kappa(L, M)$ are computed as follows. For each R -module X , factor its DG resolutions over K as

$$P_X^K \xrightarrow{(\mathrm{id}_X)^\cdot} P_X^K / \mathfrak{U} P_X^K \xrightarrow{(\varepsilon_X^K)^\cdot} X \xrightarrow{(\eta_X^K)^\cdot} (0: \mathfrak{U})_{I_K^K} \xrightarrow{(\mathrm{id}_X)^\cdot} I_K^K.$$

Note that the intermediate complexes of R -modules are (respectively) DG-projective and DG-injective, and that the induced maps $(\varepsilon_X^K)^\cdot$ and $(\eta_X^K)^\cdot$ are quasi-isomorphisms. Then choose $\varepsilon_X^R = (\varepsilon_X^K)^\cdot$ and $\eta_X^R = (\eta_X^K)^\cdot$ as the DG resolutions of X over R .

2.2. Fundamental Sequences. We consider for each j the DG subalgebra $K[j] \subset K$ generated over $K_0 = Q$ by $\xi_1, \dots, \xi_j, \dots, \xi_c$, and the multiplication map

$$\beta_j: K \otimes_{K[j]} K \rightarrow K \quad \text{with } \beta_j(b \otimes c) = bc.$$

PROPOSITION. Let L, M, N , be K -modules.

(a) The canonical inclusion $\beta_j|_N^M$ and the degree one map $\alpha_j|_N^M$ defined by

$$\gamma \mapsto (m \mapsto (-1)^{|\gamma|} \gamma(\xi_j m) - \xi_j \gamma(m))$$

yield an exact sequence of chain maps

$$0 \rightarrow \operatorname{Hom}_K(M, N) \xrightarrow{\beta_j|_N^M} \operatorname{Hom}_{K[j]}(M, N) \xrightarrow{\alpha_j|_N^M} \operatorname{Hom}_K(M, N)$$

in which $\alpha_j|_N^M$ is surjective if M^\natural is projective or N^\natural is injective over K^\natural .

(b) The canonical projection $\beta_j|_{LM}$ and the degree one map $\alpha_j|_{LM}$ defined by

$$l \otimes_K m \mapsto (-1)^{|l|} l \otimes_{K[j]} (\xi_j m) - (\xi_j l) \otimes_{K[j]} m$$

yield an exact sequence of chain maps

$$L \otimes_K M \xrightarrow{\alpha_j|_{LM}} L \otimes_{K[j]} M \xrightarrow{\beta_j|_{LM}} L \otimes_K M \rightarrow 0$$

in which $\alpha_j|_{LM}$ is injective if L^\natural or M^\natural is flat over K^\natural .

Proof. To simplify notation, we set $A = K[j]$ and $\xi = \xi_j$.

A direct computation yields equalities

$$\operatorname{Ker} \beta_j = (K \otimes_A K)(1 \otimes_A \xi - \xi \otimes_A 1) = (0:_{K \otimes_A K} (1 \otimes_A \xi - \xi \otimes_A 1)).$$

Thus, left multiplication by $1 \otimes_A \xi - \xi \otimes_A 1$ on $K \otimes_A K$ defines a degree 1 homomorphism of graded $K \otimes_A K$ -modules

$$\alpha_j: K \rightarrow K \otimes_A K \quad \text{given by} \quad a \mapsto 1 \otimes_A (\xi a) - \xi \otimes_A a$$

which fits into a fundamental exact sequence

$$0 \rightarrow K \xrightarrow{\alpha_j} K \otimes_A K \xrightarrow{\beta_j} K \rightarrow 0.$$

More verifications show that the action $((b \otimes_A c)\gamma)(m) = (-1)^{|c||\gamma|} b\gamma(cm)$ gives $\operatorname{Hom}_A(M, N)$ a structure of $K \otimes_A K$ -module, and that the action $(b \otimes_A c)(l \otimes_A m) = (-1)^{|c||l|} bl \otimes_A cm$ produces such a structure for $L \otimes_A M$. Furthermore, if X is a $K \otimes_A K$ -module and K acts on $X \otimes_K M$ by $a(x \otimes m) = (ax) \otimes m$, then there is a canonical

isomorphism

$$\mathrm{Hom}_{(K \otimes_A K)}(X, \mathrm{Hom}_A(M, N)) \cong \mathrm{Hom}_K(X \otimes_K M, N)$$

given by $\gamma \mapsto ((x \otimes_K m) \mapsto \gamma(x)(m))$, and a canonical isomorphism

$$(L \otimes_A M) \otimes_{(K \otimes_A K)} X \cong L \otimes_K (X \otimes_K M)$$

given by $(l \otimes m) \otimes x \mapsto (-1)^{|m||x|} l \otimes (x \otimes m)$. To obtain the exact sequence in (a), apply the functor $\mathrm{Hom}_{(K \otimes_A K)}(-, \mathrm{Hom}_A(M, N))$ to the fundamental exact sequence, and set $\alpha_j|_N^M = \mathrm{Hom}(\alpha_j, \mathrm{Hom}(M, N))$. For the sequence in (b), apply $(L \otimes_A M) \otimes_{(K \otimes_A K)} -$ to the fundamental sequence, and set $\alpha_j|_{LM} = (L \otimes M) \otimes \alpha_j$. Finally, use the isomorphisms above with $X = K$ to identify the outer terms.

The first isomorphism shows that the sequence in (a) can be obtained in two stages. The first is an application of $- \otimes_K M$ to the fundamental exact sequence, with the K -module structure on $K \otimes_A K$ given by $a(b \otimes c) = (-1)^{|a||b|} b \otimes (ac)$. The map β_j^\natural is then split by $a \mapsto 1 \otimes a$, so we get an exact sequence of K -modules

$$0 \rightarrow M \xrightarrow{\alpha_j|_{KM}} K \otimes_A M \xrightarrow{\beta_j|_{KM}} M \rightarrow 0.$$

Thus, $\alpha_j|_N^M$ is surjective if N^\natural is injective or M^\natural is projective over K^\natural .

A similar argument proves the injectivity of $\alpha_j|_{LM}$ if L^\natural or M^\natural is flat over K^\natural . ■

2.3. Cohomology Operators. Let $\mathbf{x} = \{x_1, \dots, x_c\} \subset Q$ be a Koszul-regular set. Let L, M, N , be complexes of R -modules. We view them as K -modules via κ and introduce shorthand notation for their resolutions over K

$$U = P_L^K, \quad V = P_M^K, \quad Y = I_K^N.$$

Then we choose as in Subsection 2.1 resolutions over R

$$E = U/\mathfrak{A}U = P_L^R, \quad F = V/\mathfrak{A}V = P_M^R, \quad J = (0:\mathfrak{A})_Y = I_R^N,$$

and let $\varepsilon_U: U \rightarrow E$, $\varepsilon_V: V \rightarrow F$, and $\eta^Y: J \rightarrow Y$ be the canonical maps.

By Subsection 2.2 there are exact sequences of K -modules

$$0 \rightarrow \mathrm{Hom}_K(V, Y) \xrightarrow{\beta_j|_Y^K} \mathrm{Hom}_{K[j]}(V, Y) \xrightarrow{\alpha_j|_Y^K} \mathrm{Hom}_K(V, Y) \rightarrow 0$$

and

$$0 \rightarrow U \otimes_K V \xrightarrow{\alpha_j|_{UV}} U \otimes_{K[j]} V \xrightarrow{\beta_j|_{UV}} U \otimes_K V \rightarrow 0.$$

The connecting homomorphisms ∂_j of the associated long (co)homology exact sequences define operators χ_j by commutativity of the squares

$$\begin{array}{ccc} \mathrm{Ext}_R^i(M, N) & \xrightarrow{\cong} & \mathrm{Ext}_K^i(M, N) \\ \chi_j \downarrow & & \downarrow \partial_j \\ \mathrm{Ext}_R^{i+2}(M, N) & \xrightarrow{\cong} & \mathrm{Ext}_K^{i+2}(M, N) \end{array} \quad \begin{array}{ccc} \mathrm{Tor}_i^K(L, M) & \xrightarrow{\cong} & \mathrm{Tor}_i^R(L, M) \\ \partial_j \downarrow & & \downarrow \chi_j \\ \mathrm{Tor}_{i-2}^K(L, M) & \xrightarrow{\cong} & \mathrm{Tor}_{i-2}^R(L, M) \end{array}$$

where the isomorphisms are the maps $\mathrm{Ext}_K^*(M, N)$ and $\mathrm{Tor}_K^*(L, M)$ induced by

$$\mathrm{Hom}_R(F, J) \xrightarrow{\cong} \mathrm{Hom}_K(V, Y) \quad \text{with } \gamma \mapsto \eta^Y \circ \gamma \circ \varepsilon_V$$

and

$$U \otimes_K V \xrightarrow{\cong} E \otimes_R F \quad \text{with } u \otimes_K v \mapsto \varepsilon_U(u) \otimes_R \varepsilon_V(v).$$

We call $\chi = (\chi_j)_1^c$ the family of cohomology operators defined by \mathbf{x} .

For another construction of cohomology operators we study

2.4. DG-Projectives. Let P be a DG-projective K -module.

The projectivity of P^\natural over K^\natural , cf. Subsection 1.2, implies that of $\bar{P} = P^\natural / K_+^\natural P^\natural$ over Q , hence the epimorphism $P^\natural \rightarrow \bar{P}$ is split by a homomorphism σ of graded Q -modules. The degree zero homomorphism of K^\natural -modules $\pi: K^\natural \otimes_Q \mathrm{Im}(\sigma) \rightarrow P^\natural$ given by $\pi(a \otimes p) = ap$ induces an isomorphism $\pi \otimes_{K^\natural} Q$, so $\mathrm{Coker}(\pi) = 0$. Thus, we have an exact sequence of graded K^\natural -modules

$$0 \rightarrow \mathrm{Ker} \pi \rightarrow K^\natural \otimes_Q \mathrm{Im}(\sigma) \xrightarrow{\pi} P^\natural \rightarrow 0$$

which splits because P^\natural is projective. In the exact sequence obtained by applying $- \otimes_{K^\natural} Q$ the map $\pi \otimes_{K^\natural} Q$ is bijective, hence $\mathrm{Ker}(\pi) \otimes_{K^\natural} Q = 0$. Due to the nilpotency of the ideal $K_+^\natural = (\xi_1, \dots, \xi_c)K^\natural$, we have $\mathrm{Ker}(\pi) = 0$, hence π is bijective.

For $\{h_1, \dots, h_i\} = H \subseteq Z = \{1, \dots, c\}$ with $h_1 < \dots < h_i$, set $\xi_H = \xi_{h_1} \wedge \dots \wedge \xi_{h_i} \in K$. Each $v \in P$ can be written in the form $v = \sum_{H \subseteq Z} \xi_H \langle v \rangle_H$ with uniquely defined $\langle v \rangle_H \in \mathrm{Im}(\sigma)$, so for each $H \subseteq Z$ we have a Q -linear endomorphism

$$\partial_H: \mathrm{Im}(\sigma) \rightarrow \mathrm{Im}(\sigma) \quad \text{with } \partial_H(p) = (-1)^{|H|} \langle \partial p \rangle_H$$

of degree $(-|H| - 1)$. The equation $\partial^2(p) = 0$ then yields

$$\partial_{\mathcal{O}}^2(p) = \sum_{j=1}^c x_j \partial_j(p); \quad (2.4.1)$$

$$\partial_{\mathcal{O}} \partial_j(p) \equiv \partial_j \partial_{\mathcal{O}}(p) \pmod{\alpha \operatorname{Im}(\sigma)} \quad \text{for } j = 1, \dots, c. \quad (2.4.2)$$

Note that $D = R \otimes_K P \simeq \operatorname{Im}(\sigma)/\alpha \operatorname{Im}(\sigma) = P/\mathfrak{A}(P)$ is a complex of projective R -modules with differential $\partial(1 \otimes p) = 1 \otimes \partial_{\mathcal{O}}(p)$, that the maps

$$\tau_j^D: D \rightarrow D \quad \text{given by } 1 \otimes p \mapsto 1 \otimes \partial_j(p) \text{ for } j = 1, \dots, c$$

are degree -2 chain endomorphisms of D , and that the family $\tau^D = (\tau_j^D)_1^c$ is uniquely determined by \mathbf{x} . It allows for chain level computations of the operators.

PROPOSITION. *Let L, M, N , be complexes of R -modules, and let $\varepsilon_L^R: E \rightarrow L$, $\varepsilon_M^R: F \rightarrow M$, $\eta_R^M: M \rightarrow I$, be their DG resolutions over R described in Subsection 2.3.*

If $\tau_1^E, \dots, \tau_c^E: E \rightarrow E$ and $\tau_1^F, \dots, \tau_c^F: F \rightarrow F$ are families of chain endomorphisms determined by \mathbf{x} , then for $j = 1, \dots, c$ there are equalities

$$\chi_j = \operatorname{H} \operatorname{Hom}_R(\tau_j^F, J): \operatorname{Ext}_R^*(M, N) \rightarrow \operatorname{Ext}_R^*(M, N);$$

$$\chi_j = \operatorname{H}(\tau_j^E \otimes_R F): \operatorname{Tor}_*^R(L, M) \rightarrow \operatorname{Tor}_*^R(L, M);$$

$$\chi_j = -\operatorname{H}(E \otimes_R \tau_j^F): \operatorname{Tor}_*^R(L, M) \rightarrow \operatorname{Tor}_*^R(L, M).$$

Proof. Along with the chosen in Subsection 2.3 DG-projective resolutions V and F of M over K and R , we consider the complex of R -modules $F[j] = R \otimes_{K[j]} V$. Identifying $F[j]^{\natural}$ with $F^{\natural} \otimes \xi_j F^{\natural}$, we see that its differential is given by

$$\partial(f' + \xi_j f'') = \partial(f') - \xi_j(\tau_j^F(f') + \partial(f'')) \quad \text{for } f', f'' \in F.$$

Due to Subsection 2.2 we have a commutative diagram of chain maps with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & V & \xrightarrow{\alpha_j|_{KV}} & K \otimes_{K[j]} V & \xrightarrow{\beta_j|_{KV}} & V \longrightarrow 0 \\ & & \cong \downarrow & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & F & \xrightarrow{\ddot{\alpha}_j|_{RF}} & F[j] & \xrightarrow{\ddot{\beta}_j|_{RF}} & F \longrightarrow 0 \end{array} \quad (2.4.3)$$

where the middle vertical arrow is $\kappa \otimes_{K[j]} V$ and the maps in the bottom row are $\ddot{\alpha}_j|_{RF}(f) = \xi_j f$ and $\ddot{\beta}_j|_{RF}(f' + \xi_j f'') = f'$.

Apply $\text{Hom}_K(-, Y)$ to (2.4.3). As Y is DG-injective over K , this functor is exact. Furthermore, on complexes of R -modules it reduces to $\text{Hom}_R(-, J)$, so we get a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & \text{Hom}_K(V, Y) & \xrightarrow{\beta_j|_Y^V} & \text{Hom}_{K[j]}(V, Y) & \xrightarrow{\alpha_j|_Y^V} & \text{Hom}_K(V, Y) \longrightarrow 0 \\ & \cong \uparrow & & \uparrow \cong & & \uparrow \cong \\ 0 \longrightarrow & \text{Hom}_R(F, J) & \xrightarrow{\ddot{\beta}_j|_J^F} & \text{Hom}_R(F[j], J) & \xrightarrow{\ddot{\alpha}_j|_J^F} & \text{Hom}_R(F, J) \longrightarrow 0 \end{array}$$

and chain maps $\ddot{\beta}_j|_J^F(\tilde{\gamma})(f' + \xi_j f'') = \tilde{\gamma}(f')$ and $\ddot{\alpha}_j|_J^F(\tilde{\gamma})(f) = (-1)^{|\gamma|} \tilde{\gamma}(\xi_j f)$. Its commutativity shows that the operator χ_j on $\text{Ext}_R^*(M, N)$ can be computed by the connecting homomorphism $\ddot{\beta}_j$ of the lower row. A direct computation shows that if γ is the class of a chain map $\tilde{\gamma}: F \rightarrow J$, then $\ddot{\beta}_j(\gamma) = [\tilde{\gamma} \circ \tau_j^F]$. In other words, $\chi_j(\gamma) = \text{H Hom}_R(\tau_j^F, J)(\gamma)$.

Similar considerations applied to the DG-projective resolutions U and E of L over K and R produce a commutative diagram of chain maps with exact rows complex

$$\begin{array}{ccccccc} 0 \longrightarrow & E & \xrightarrow{\dot{\alpha}_j|_{ER}} & E[j] & \xrightarrow{\dot{\beta}_j|_{ER}} & E \longrightarrow 0 \\ & \cong \uparrow & & \uparrow \cong & & \uparrow \cong \\ 0 \longrightarrow & U & \xrightarrow{\alpha_j|_{UK}} & U \otimes_{K[j]} K & \xrightarrow{\beta_j|_{UK}} & U \longrightarrow 0 \end{array} \quad (2.4.4)$$

with $E[j] = E \otimes_{K[j]} R$. In it the middle quasi-isomorphism is $U \otimes_{K[j]} \kappa$, and the maps in the top row are $\dot{\alpha}_j|_{ER}(e) = -\xi_j e$ and $\dot{\beta}_j|_{ER}(e' + \xi_j e'') = e'$.

Applying $(- \otimes_K V)$ to (2.4.4) and $(U \otimes_K -)$ to (2.4.2), we arrive at the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & E \otimes_R F & \xrightarrow{\dot{\alpha}_j|_{EF}} & E[j] \otimes_R F & \xrightarrow{\dot{\beta}_j|_{EF}} & E \otimes_R F \longrightarrow 0 \\ & \cong \uparrow & & \uparrow \cong & & \uparrow \cong \\ 0 \longrightarrow & U \otimes_R V & \xrightarrow{\alpha_j|_{UV}} & U \otimes_{K[j]} V & \xrightarrow{\beta_j|_{UV}} & U \otimes_R V \longrightarrow 0 \\ & \cong \downarrow & & \downarrow \cong & & \downarrow \cong \\ 0 \longrightarrow & E \otimes_R F & \xrightarrow{\ddot{\alpha}_j|_{EF}} & E \otimes_R F[j] & \xrightarrow{\ddot{\beta}_j|_{EF}} & E \otimes_R F \longrightarrow 0 \end{array}$$

which is commutative with exact rows. By the bottom portion, χ_j on $\text{Tor}_*^R(L, M)$ is the connecting homomorphism $\dot{\beta}_j$ of a short exact sequence of complexes in which $\ddot{\alpha}_j|_{EF}(e \otimes f) = (-1)^{|e|} e \otimes (\xi_j f)$ and $\ddot{\beta}_j|_{EF}(e \otimes (f' + \xi_j f'')) = e \otimes f'$. An easy calculation yields $\dot{\beta}_j([\sum_k e_k \otimes f_k]) = -[\sum_k e_k \otimes \tau_j^F(f_k)]$, hence $\chi_j = -\text{H}(F \otimes_R \tau_j^F)$.

The top part shows that the action of χ_j on $\text{Tor}_*^R(L, M)$ is also given by the connecting homomorphism ∂_j of an exact sequence $\dot{\alpha}_j|_{EF}(e \otimes f) = -(\xi_j e) \otimes f$, and $\dot{\beta}_j|_{EF}((e' + \xi_j e'') \otimes f) = e' \otimes f$. This time, $\partial_j([\sum_k e_k \otimes f_k]) = [\sum_k \tau_j^E(e_k) \otimes f_k]$. ■

COROLLARY. *If $\eta_R^M: M \rightarrow I$ is a DG-injective resolution over R and $1_M = [\eta_R^M \circ \varepsilon_M^R] \in \text{Ext}_R^0(M, M)$, then $\chi_j(1_M) = [\eta_R^M \circ \varepsilon_M^T \circ \tau_j^F] \in \text{Ext}_R^2(M, M)$ for $j = 1, \dots, c$.*

An alternative approach to operators in cohomology is obtained by using

2.5. DG-injectives. In addition to the assumptions and notation of Subsections 2.1 and 2.3, let $\{\xi_H^*\}_{H \subseteq Z}$ be the Q -basis of $\text{Hom}_Q(K^\natural, Q)$ dual to the basis $\{\xi_H\}_{H \subseteq Z}$ of K^\natural . The action of K^\natural on $\text{Hom}_Q(K^\natural, Q)$ is described by $\xi_j \xi_H^* = 0$ if $j \notin H$, and $\xi_j \xi_H^* = (-1)^{r-1} \xi_{H \setminus j}^*$ if $j = h_r$, where $H = \{h_1, \dots, h_i\}$ and $h_1 < \dots < h_i$.

Let I be a DG-injective K -module.

The graded module $I = (0: K_+^\natural)_I$ is then injective over Q , due to the injectivity of I^\natural over K^\natural , cf. Subsection 1.2. Let ρ be a Q -linear splitting of $\underline{I} \subseteq I^\natural$, and set $\iota(y)(a) = (-1)^{|y||a|} \rho(ay)$ for $y \in I^\natural$ and $a \in K^\natural$. Therefore, $\iota: I^\natural \rightarrow \text{Hom}_Q(K^\natural, I)$ is a degree 0 homomorphism of graded K^\natural -modules. As $\text{Hom}_{K^\natural}(Q, \iota)$ is bijective, we see that $\text{Ker}(\iota) = 0$, so ι is injective and yields an exact sequence of graded K^\natural -modules

$$0 \rightarrow I^\natural \xrightarrow{\iota} \text{Hom}_Q(K^\natural, \underline{I}) \rightarrow \text{Coker } \iota \rightarrow 0.$$

The sequence splits, due to the injectivity of I^\natural . Applying to it $\text{Hom}_{K^\natural}(Q, -)$ we get an exact sequence whose first map is bijective, so $\text{Hom}_{K^\natural}(Q, \text{Coker } \iota) = 0$. We remark that if X is a K^\natural -module such that $\text{Hom}_{K^\natural}(Q, X) = 0$, then the nilpotency of the graded ideal K_+^\natural implies that $X = 0$. Thus, $\text{Coker } \iota = 0$ and ι is bijective.

The map $\kappa: K^\natural \otimes_Q \underline{I} \rightarrow \text{Hom}_Q(K^\natural, Q) \otimes_Q \underline{I}$ with $a \otimes w \mapsto (-1)^{c|a|} a \xi_Z^* \otimes w$ is a bijective degree $(-c)$ homomorphism of graded K^\natural -modules. Thus, $y \in I$ has a unique expression of the form $y = \sum_{H \subseteq Z} \xi_H \langle y \rangle_H$ with $\langle y \rangle_H \in W = \iota^{-1} \kappa(1 \otimes_Q \underline{I})$. In particular, for each $y \in \underline{I}$ there is a unique $w \in W$ such that $y = \xi_Z w$. For $H \subseteq Z$ we define a Q -linear degree $(-|H| - 1)$ endomorphism

$$\partial_H: \underline{I} \rightarrow \underline{I} \quad \text{with } \partial_H(y) = (-1)^{c-|H|} \xi_Z \langle \partial w \rangle_H.$$

By a direct computation, the equation $\partial^2(w) = 0$ now yields a congruence

$$\langle \partial \langle \partial w \rangle_{\emptyset} \rangle_{\emptyset} \equiv - \sum_{j=1}^c x_j \langle \partial w \rangle_j + \sum_{j=1}^c \xi_j (\langle \partial \langle \partial w \rangle_j \rangle_{\emptyset} - \langle \partial \langle \partial w \rangle_{\emptyset} \rangle_j) \\ \text{mod} \left(\sum_{\{h,j\} \subseteq Z} \xi_j \alpha \langle \partial w \rangle_{\{h,j\}} + \sum_{|H| \geq 2} \xi_H W \right).$$

Multiplication with ξ_Z on the left with ξ_Z gives $\xi_Z \langle \partial \langle \partial w \rangle_{\emptyset} \rangle_{\emptyset} = -\xi_Z \sum_{j=1}^c x_j \langle \partial w \rangle_j$. Thus,

$$\partial_{\emptyset}^2(y) = (-1)^c \sum_{j=1}^c x_j \partial_j(y)$$

and the congruence above simplifies to

$$\sum_{j=1}^c \xi_j (\langle \partial \langle \partial w \rangle_{\emptyset} \rangle_j - \langle \partial \langle \partial w \rangle_j \rangle_{\emptyset}) \equiv 0 \\ \text{mod} \left(\sum_{\{h,j\} \subseteq Z} \xi_j \alpha \langle \partial w \rangle_{\{h,j\}} + \sum_{|H| \geq 2} \xi_H W \right).$$

Multiplying the new congruence on the left with $\xi_{Z \setminus j}$ for $j = 1, \dots, c$, we get

$$\partial_j \partial_{\emptyset}(y) \equiv \partial_{\emptyset} \partial_j(y) \quad \text{mod} \left(\sum_{h=1}^c \alpha \langle \partial w \rangle_{\{h,j\}} \right).$$

Remark that y is in $B = (0: \alpha)_I = (0: \mathfrak{A})_I$ if and only if w is in $(0: \alpha)_W$, and that the restriction of ∂_{\emptyset} to B is equal to the differential ∂ of B . Thus, setting $v_j = (-1)^c \partial_j$ and denoting v_j^B the restriction of v_j to B , we obtain a family $\nu^B = (v_j^B)_1^c$ of chain endomorphisms of degree -2 of the complex of R -modules B .

PROPOSITION. *Let M and N be complexes of R -modules, let $F = V/\mathfrak{A}V$ and $J = (0: \mathfrak{A})_Y$ be their DG resolutions as in Subsection 2.3, and let $v_1^J, \dots, v_c^J: J \rightarrow J$ be the family of chain endomorphisms described above. For $j = 1, \dots, c$ there are equalities*

$$\chi_j = \text{H Hom}_R(F, v_j^J): \text{Ext}_R^*(M, N) \rightarrow \text{Ext}_R^*(M, N).$$

Proof. Consider the complex of R -modules $J[j] = \text{Hom}_{K[j]}(R, Y)$, and identify $J[j]^{\natural}$ with $\xi_Z(0: \alpha)_W^{\natural} \otimes \xi_{Z \setminus j}(0: \alpha)_W^{\natural}$. The differential of $J[j]$ is then given by

$$\partial(\xi_Z w' + \xi_{Z \setminus j} w'') = \partial_j(\xi_Z w') + (-1)^j v_j(\xi_Z w'') + (-1)^{c-1} \xi_{Z \setminus j} \langle \partial w'' \rangle_{\emptyset}$$

for $w', w'' \in (0 : \alpha)_{W'}^b$. Thus, from Subsection 2.2 we get a commutative diagram of chain maps

$$\begin{array}{ccccccc}
 0 & \longrightarrow & J & \xrightarrow{\dot{\beta}_j|_J^R} & J[j] & \xrightarrow{\dot{\alpha}_j|_J^R} & J \longrightarrow 0 \\
 & & \cong \downarrow & & \downarrow \cong & & \downarrow \cong \\
 0 & \longrightarrow & Y & \xrightarrow{\beta_j|_Y^K} & \text{Hom}_{K[j]}(K, Y) & \xrightarrow{\alpha_j|_Y^K} & Y \longrightarrow 0
 \end{array}$$

in which the rows are exact, the middle vertical arrow is induced by $\text{Hom}_{K[j]}(\kappa, Y)$ and is a quasi-isomorphism because the external two arrows are, and where

$$\dot{\beta}_j|_J^R(\xi_Z w) = \xi_Z w \quad \text{and} \quad \ddot{\alpha}_j|_J^R(\xi_Z w' + \xi_{Z \setminus j} w'') = (-1)^j \xi_Z w''.$$

The rest of the argument is similar to that for Proposition 2.4. ■

3. PROPERTIES

We derive the main formal properties of cohomology operators.

3.1. *Naturality.* Given a commutative diagram of ring homomorphisms

$$\begin{array}{ccc}
 Q' & \xrightarrow{\psi} & Q \\
 \rho' \downarrow & & \downarrow \rho \\
 R' & \xrightarrow{\varphi} & R
 \end{array}$$

with $\alpha' = \text{Ker}(\rho')$ generated by a Koszul-regular set $\mathbf{x}' = x'_1, \dots, x'_{c'} \subset Q'$, let $\chi' = \{\chi'_1, \dots, \chi'_{c'}\}$ be the family of cohomology operators defined by \mathbf{x}' , and let

$$\psi(x'_i) = \sum_{j=1}^c q_{ij} x_j \quad \text{with } q_{ij} \in Q \text{ for } 1 \leq i \leq c'$$

be equalities resulting from the inclusion $\psi(\mathbf{x}') \subset \text{Ker}(\rho)$.

We can now describe the naturality of the cohomology operators.

THEOREM. *Let $\mu: M' \rightarrow M$ be a φ -covariant chain map.*

If $\lambda: L' \rightarrow L$ is a φ -covariant chain map, then

$$\chi_j \circ \text{Tor}_*^\varphi(\lambda, \mu) = \sum_{i=1}^{c'} q_{ij} \text{Tor}_*^\varphi(\lambda, \mu) \circ \chi'_i \quad \text{for } j = 1, \dots, c.$$

If $\nu: N \rightarrow N'$ is a φ -contravariant chain map and $q_{ij} = \psi(q'_{ij})$, then

$$\sum_{i=1}^{c'} q'_{ij} \chi'_i \circ \text{Ext}^*_{\varphi}(\mu, \nu) = \text{Ext}^*_{\varphi}(\mu, \nu) \circ \chi_j \quad \text{for } j = 1, \dots, c.$$

Proof. This is immediate from Subsection 1.2, Proposition 2.4, and the following lemma. ■

LEMMA. Let K' be the Koszul complex $K(x', Q')$, and let $\Psi: K' \rightarrow K$ be the homomorphism of DG algebras defined by $\Psi(\xi'_i) = \sum_{j=1}^c q_{ij} \xi_j$ for $1 \leq i \leq c'$. Let P' be a DG-projective module over K' , and let $\tau' = (\tau'_i)_{i=1}^{c'}$ be the family of chain endomorphisms of the complex of R' -modules $D' = P'/\mathcal{A}'P'$ defined by \mathbf{x}' . Given a Ψ -covariant chain map $\gamma: P' \rightarrow P$, let $1 \otimes \gamma: D' \rightarrow D$ be the induced covariant chain map of complexes over the ring homomorphism $\varphi: R' \rightarrow R$.

The chain maps $\tau_j^D \circ (1 \otimes \gamma)$ and $\sum_{i=1}^{c'} q_{ij} (1 \otimes \gamma) \circ \tau_i^{D'}$ are homotopic.

Remark. In view of Proposition 4.2 the lemma is equivalent to [7, Proposition 1.7].

Proof. In the notation of Subsection 2.4, for each $H \subseteq Z = \{1, \dots, c\}$ consider the degree $(|\gamma| - |H|)$ homomorphisms of graded Q' -modules

$$\gamma_H: \text{Im}(\sigma') \rightarrow \text{Im}(\sigma) \quad \text{with } \gamma_H(y') = (-1)^{|H|} \langle \gamma(y') \rangle_H.$$

By definition, we have

$$\begin{aligned} \partial \gamma(y') &\equiv \partial_{\varnothing} \gamma_{\varnothing}(y') - \sum_{j=1}^c x_j \gamma_j(y') \\ &\quad + \sum_{j=1}^c \xi_j \left(\partial_{\varnothing} \gamma_j(y') - \partial_j \gamma_{\varnothing}(y') + \sum_{\{h, j\} \subseteq Z} \pm x_h \gamma_{\{h, j\}}(y') \right) \end{aligned}$$

modulo $(\sum_{|H| \geq 2} \xi_H P)$. Due to the Ψ -covariance of γ , we get

$$\begin{aligned} \gamma \partial'(y') &\equiv \gamma_{\varnothing} \partial'_{\varnothing}(y') - \sum_{j=1}^c \xi_j \gamma_j \partial'_{\varnothing}(y') - (-1)^{|\gamma|} \sum_{i=1}^{c'} \Psi(\xi'_i) \gamma_{\varnothing} \partial'_i(y') \\ &= \gamma_{\varnothing} \partial'_{\varnothing}(y') + \sum_{j=1}^c \xi_j \left((-1)^{|\gamma|-1} \sum_{i=1}^{c'} q_{ij} \gamma_{\varnothing} \partial'_i(y') - \gamma_j \partial'_{\varnothing}(y') \right) \end{aligned}$$

modulo $(\sum_{|H| \geq 2} \xi_H P)$. Noting that $\partial \gamma = (-1)^{|\gamma|} \gamma \partial'$, and that γ_{\varnothing} induces the morphism $1 \otimes \gamma: D' \rightarrow D$, we conclude by comparing the coefficients

of ξ_j that

$$\tau_j^D \circ (1 \otimes \gamma) - \sum_{i=1}^{c'} q_{ij}(1 \otimes \gamma) \circ \tau_i^{D'} = \partial \gamma_j - (-1)^{|\gamma_j|} \gamma_j \partial'$$

for $j = 1, \dots, c$.

Thus, γ_j is a homotopy from $\tau_j^D \circ (1 \otimes \gamma)$ to $\sum_{i=1}^{c'} q_{ij}(1 \otimes \gamma) \circ \tau_i^{D'}$. ■

3.2. Centrality. Cohomology operators (anti)commute with products:

THEOREM. For $j = 1, \dots, c$ there are equalities

$$\begin{aligned}\chi_j(\gamma) \cdot \delta &= \chi_j(\gamma \cdot \delta) = \gamma \cdot \chi_j(\delta) \in \text{Ext}_R^*(L, N) \\ \chi_j(\gamma) \cdot t &= -\chi_j(\gamma \cdot t) = -\gamma \cdot \chi_j(t) \in \text{Tor}_*^R(L, N) \\ \chi_j(\delta) \cdot u &= \chi_j(\delta \cdot u) = \delta \cdot \chi_j(u) \in \text{Tor}_*^R(M, N)\end{aligned}$$

whenever $\gamma \in \text{Ext}_R^*(M, N)$, $\delta \in \text{Ext}_R^*(L, M)$, $t \in \text{Tor}_*^R(L, M)$, $u \in \text{Tor}_*^R(L, N)$.

Proof. In addition to the resolutions F, E, J , described in Subsection 2.3, we consider DG resolutions $M \xrightarrow{\sim} I_{R_\infty}^M = I$ and $G = P_N^R \xrightarrow{\sim} N$.

Choose chain maps $\tilde{\gamma}: F \rightarrow J$ and $\tilde{\delta}: E \rightarrow I$ such that $\gamma = [\tilde{\gamma}]$ and $\delta = [\tilde{\delta}]$, cf. Subsection 1.2. Extend $\tilde{\gamma}$ to $\tilde{\gamma} \in \text{Hom}_R(I, J)$ and lift $\tilde{\delta}$ to $\tilde{\delta} \in \text{Hom}_R(E, F)$. Lemma 3.1 applied to id_K shows that $\tau_j^F \circ \tilde{\delta} \cdot$ and $\tilde{\delta} \cdot \circ \tau_j^E$ are homotopic, so Proposition 2.4 gives

$$\begin{aligned}\chi_j(\gamma) \cdot \delta &= [\tilde{\gamma} \circ \tau_j^F] \cdot \delta = [\tilde{\gamma} \circ (\tau_j^F \circ \tilde{\delta} \cdot)] = [\tilde{\gamma} \circ (\tilde{\delta} \cdot \circ \tau_j^E)] \\ &= [(\tilde{\gamma} \circ \tilde{\delta} \cdot) \circ \tau_j^E] = \chi_j(\gamma \cdot \delta).\end{aligned}$$

On the other hand, by [6, Sect. 7, No. 2] and Proposition 2.4 we have

$$\begin{aligned}\chi_j(\gamma \cdot \delta) &= \chi_j([\tilde{\gamma} \circ \tilde{\delta} \cdot]) = [(\tilde{\gamma} \circ \tilde{\delta} \cdot) \circ \tau_j^E] = [(\tilde{\gamma} \circ \tilde{\delta}) \circ \tau_j^E] \\ &= \gamma \cdot [\tilde{\delta} \cdot \circ \tau_j^E] = \gamma \cdot \chi_j(\delta).\end{aligned}$$

With $\sum_k e_k \otimes f_k$ representing t and $\tilde{\gamma}: F \rightarrow G$ lifting $\tilde{\gamma}$, Proposition 2.4 yields

$$\begin{aligned}\chi_j(\gamma) \cdot t &= [\gamma \circ \tau_j^F] \cdot \left[\sum_k e_k \otimes f_k \right] \\ &= \left[\sum_k (-1)^{(|\gamma|-2)|e_k|} e_k \otimes \tilde{\gamma}(\tau_j^F(f_k)) \right]; \\ \chi_j(\gamma \cdot t) &= \chi_j \left(\left[\sum_k (-1)^{|\gamma||e_k|} e_k \otimes \tilde{\gamma}(f_k) \right] \right) \\ &= - \left[\sum_k (-1)^{|\gamma||e_k|} e_k \otimes \tau_j^G(\tilde{\gamma}(f_k)) \right]; \\ \gamma \cdot \chi_j(t) &= \gamma \cdot \left[- \sum_k e_k \otimes (\tau_j^F(f_k)) \right] \\ &= - \left[\sum_k (-1)^{|\gamma||e_k|} e_k \otimes \tilde{\gamma}(\tau_j^F(e_k)) \right].\end{aligned}$$

As $\tau_j^G \circ \tilde{\gamma}$ and $\tilde{\gamma} \circ \tau_j^F$ are homotopic by Lemma 3.1, we get the second statement.

Finally, if $\sum_k e_k \otimes g_k$ represents u , then by Proposition 2.4 we have

$$\begin{aligned}\chi_j(\delta) \cdot u &= [\tilde{\delta} \circ \tau_j^E] \cdot \left[\sum_k e_k \otimes g_k \right] = \left[\sum_k (\tilde{\delta} \circ \tau_j^E(e_k)) \otimes g_k \right]; \\ \chi_j(\delta \cdot u) &= \chi_j \left(\left[\sum_k (\tilde{\delta}(e_k)) \otimes g_k \right] \right) = \left[\sum_k (\tau_j^F \circ \tilde{\delta}(e_k)) \otimes g_k \right]; \\ \delta \cdot \chi_j(u) &= \delta \cdot \left[\sum_k (\tau_j^E(e_k)) \otimes g_k \right] = \left[\sum_k (\tilde{\delta} \circ \tau_j^E(e_k)) \otimes g_k \right].\end{aligned}$$

As $\tau_j^F \circ \tilde{\delta}$ and $\tilde{\delta} \circ \tau_j^E$ are homotopic, we have the last statement. ■

2.7. Characteristic Homomorphism. Let $C^* = \text{Hom}_R(C, R)$ be the dual module of $C = \alpha/\alpha^2$, and let x_1^*, \dots, x_c^* denote the basis of $C^* = (\alpha/\alpha^2)^*$ dual to the basis $\bar{\mathbf{x}} = \{x_1 + \alpha^2, \dots, x_c + \alpha^2\}$ of C .

THEOREM. *The unique homomorphism of R -modules*

$$C^* \rightarrow \text{Ext}_R^2(M, M) \quad \text{with } x_j^* \mapsto \chi_j(1_M) \text{ for } j = 1, \dots, c$$

is independent of the choice of the Koszul-regular set \mathbf{x} . Its image lies in the center of $\text{Ext}_R^(M, M)$, and so it extends canonically to a characteristic*

homomorphism of graded R -algebras from the symmetric algebra $\mathcal{S} = \text{Sym}_R(C^*)$:

$$\zeta_M: \mathcal{S} \rightarrow \text{Ext}_R^*(M, M).$$

The maps ζ_M and ζ_N induce on $\text{Ext}_R^*(M, N)$ the same \mathcal{S} -module structure,

$$\zeta_N(x_j^*) \cdot \gamma = \chi_j(\gamma) = \gamma \cdot \zeta_M(x_j^*) \quad \text{for } \gamma \in \text{Ext}_R^*(M, N).$$

The maps ζ_L and ζ_M induce on $\text{Tor}_*^R(L, M)$ opposite \mathcal{S} -module structures,

$$\zeta_L(x_j^*) \cdot t = \chi_j(t) = -\zeta_M(x_j^*) \cdot t \quad \text{for } t \in \text{Tor}_*^R(L, M).$$

COROLLARY. On $\text{Ext}_R^*(M, N)$ and $\text{Tor}_*^R(L, M)$ there are equalities

$$\chi_i \chi_j = \chi_j \chi_i \quad \text{for } 1 \leq i, j \leq c.$$

Remark. When M is an R -module the characteristic homomorphism ζ_M coincides with one constructed by Mehta, cf. Subsection 4.4; his definition does not seem to generalize to complexes of R -modules. When $c = 1$, he proves in [9, Proposition 2.3] that the image of ζ_M is central and in [9, Proposition 2.4] that over complete local rings the actions on Tor coincide. The theorem extends (and corrects some signs of) [9, Proposition 2.3].

Proof of the Theorem. Theorem 3.2 yields the formulas for the action of $\zeta_M(x_j^*)$, and the first one of them implies that the image of ζ_M is in the center of $\text{Ext}_R^*(M, M)$.

Along with \mathbf{x} , consider a Koszul-regular set $\mathbf{x}' \subset Q$ such that $(\mathbf{x}') = (\mathbf{x})$. As both $\bar{\mathbf{x}}$ and $\bar{\mathbf{x}}'$ are bases of C , the set \mathbf{x}' also consists of c elements, say x'_1, \dots, x'_c . Let $\tau_1'^F, \dots, \tau_c'^F$ be the chain endomorphisms of Subsection 2.4 defined by \mathbf{x}' on the chosen DG projective resolution F of M over R . Writing $x'_i = \sum_{j=1}^c q_{ij} x_j$, and applying Lemma 3.1 to the identity maps of K and of $P = P_M^K$, we get

$$\tau_j^F = \sum_{i=1}^c q_{ij} \tau_i'^F \quad \text{for } j = 1, \dots, c.$$

It follows that the degree zero R -linear homomorphism $(\alpha/\alpha^2)^* \rightarrow \text{Hom}_R(F, F)$ with $x_j^* \mapsto \tau_j^F$ for $j = 1, \dots, c$ does not depend on the choice of the Koszul-regular set \mathbf{x} . In view of the definition of ζ_M and Corollary 2.4 we have

$$\zeta_M(x_j^*) = [\eta_R^M \circ \varepsilon_M^R \circ \tau_j^F] = \chi_j(1_M).$$

This establishes the independence of ζ_M from the choice of \mathbf{x} . ■

4. COMPARISON

This section presents the cohomology operators constructed by Gulliksen [8], Eisenbud [7], Mehta [9], Avramov [1], and compares them to the operators χ_j from Subsection 2.3.

Throughout this section, $\rho: Q \rightarrow R$ is a surjective homomorphism of commutative rings with $\text{Ker}(\rho) = \alpha$ generated by a Koszul-regular set $\mathbf{x} = x_1, \dots, x_c$. We consider R -modules L, M, N ; as a further adjustment, we assume that DG-resolutions U, V, Y of L, M, N over K have been chosen with $U_i = 0$ and $V_i = 0$ for $i < 0$, respectively $Y_i = 0$ for $i > 0$. This can always be done, cf. [3], and has the effect that the chosen in Subsection 2.3 complexes $F = U/\mathfrak{A}U$ and $E = V/\mathfrak{A}V$ are classical projective resolutions of the R -modules L and M , while the complex $J = (0: \mathfrak{A})_Y$ is a classical injective resolution of the R -module N . Finally, recall the identifications

$$\text{Ext}_R^*(M, N) = \text{HHom}_R(F, J) \quad \text{and} \quad \text{Tor}_*^R(L, M) = \text{H}(E \otimes_R F).$$

4.1. *Gulliksen's Construction* [8, p. 176]. This starts from short exact sequences

$$0 \rightarrow E \xrightarrow{\alpha'_j} E[j] \xrightarrow{\beta'_j} E \rightarrow 0$$

and

$$0 \rightarrow F \xrightarrow{\alpha''_j} F[j] \xrightarrow{\beta''_j} F \rightarrow 0,$$

where $E[j]$ and $F[j]$ are as in Subsection 2.4; the maps in the first sequence are given by $\alpha'_j(e) = (-1)^{|e|}\xi_j e$ and $\beta'_j(e' + \xi_j e'') = e'$; those in the second sequence are defined similarly.

The preceding sequences yield exact sequences of (anti)chain¹ maps

$$\begin{aligned} 0 \rightarrow \text{Hom}_R(F, N) &\xrightarrow{\text{Hom}_R(\beta''_j, N)} \text{Hom}_R(F[j], N) \\ &\xrightarrow{\text{Hom}_R(\alpha''_j, N)} \text{Hom}_R(F, N) \rightarrow 0 \end{aligned}$$

and

$$0 \rightarrow E \otimes_R M \xrightarrow{\alpha'_j \otimes_R M} E[j] \otimes_R M \xrightarrow{\beta'_j \otimes_R M} E \otimes_R M \rightarrow 0$$

whose connecting homomorphisms we denote δ'_j .

¹ The relations $\alpha'_j \partial = \partial \alpha'_j$ and $\alpha''_j \partial = \partial \alpha''_j$ are opposite to those in Subsection 1.1 for degree 1 chain maps.

Gulliksen defines X_j for $j = 1, \dots, c$ by commutativity of the squares

$$\begin{array}{ccc} \text{Ext}_R^i(M, N) & \xleftarrow{\cong} & H_{-i} \text{Hom}_R(F, N) \\ X_j \downarrow & & (-1)^{i+j} \downarrow \\ \text{Ext}_R^{i+2}(M, N) & \xleftarrow{\cong} & H_{-i-2} \text{Hom}_R(F, N) \\ H_i(E \otimes_R M) & \xleftarrow{\cong} & \text{Tor}_i^R(L, M) \\ \downarrow (-1)^{i+j} & & \downarrow X_j \\ H_{i-2}(E \otimes_R M) & \xleftarrow{\cong} & \text{Tor}_{i-2}^R(L, M) \end{array}$$

with isomorphisms $H \text{Hom}_R(F, \eta_R^N)$ and $H(E \otimes_R \varepsilon_M^R)$.

PROPOSITION. *There are equalities $\chi_j = X_j$ for $j = 1, \dots, c$.*

Proof. Set $\omega(f) = (-1)^{|f|}f$, and paste (2.4.3) to the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F & \xrightarrow{\check{\alpha}_j|_{RF}} & F[j] & \xrightarrow{\check{\beta}_j|_{RF}} & F \longrightarrow 0 \\ & & \omega \downarrow & & \parallel & & \parallel \\ 0 & \longrightarrow & F & \xrightarrow{\alpha_j''} & F[j] & \xrightarrow{\beta_j''} & F \longrightarrow 0 \end{array}$$

along their common row. The diagram induced by applying $\text{Hom}_R(-, N)$ implies the desired assertion for $\text{Ext}_R^*(M, N)$. The argument for $\text{Tor}_*^R(L, N)$ is similar. ■

4.2. *Eisenbud's Construction* [7, Sect. 1]. This begins with a *lifting* of a complex of free R -modules (F', ∂) to a pair $(\tilde{F}', \tilde{\partial})$ of a graded free Q -module \tilde{F}' and its endomorphism $\tilde{\partial}$ of degree -1 , such that $(F' \otimes_Q R, \tilde{\partial} \otimes_Q R) \simeq (F', \partial)$. It is easy to see that liftings always exist. Fixing one, note that $\tilde{\partial}^2 \otimes_Q R = 0$, hence $\tilde{\partial}^2 = \sum_{j=1}^c x_j \tilde{t}_j(\tilde{F}', \tilde{\partial})$ for some family $(\tilde{t}_j(\tilde{F}', \tilde{\partial}))_1^c$ of degree -2 endomorphisms of the graded Q -module \tilde{F}' . As \bar{x} is a basis of the cotangent module, $\tilde{t}_j(\tilde{F}', \tilde{\partial}) \otimes_Q R$ is a chain endomorphism of $\tilde{F}' \otimes_Q R$. We denote by $t_j^{F'}$ the corresponding degree -2 chain endomorphism of F' .

For $j = 1, \dots, c$ Eisenbud defined ϵ_j^{left} by commutativity of the square

$$\begin{array}{ccc} \text{Ext}_R^i(M, N) & \xleftarrow{\cong} & H_{-i} \text{Hom}_R(F', N) \\ \epsilon_j^{\text{left}} \downarrow & & H \text{Hom}_R(t_j^{F'}, N) \downarrow \\ \text{Ext}_R^{i+2}(M, N) & \xleftarrow{\cong} & H_{-i-2} \text{Hom}_R(F', N) \end{array}$$

and $\epsilon_j^{\text{right}}$ by that of the square

$$\begin{array}{ccc} H_i(L \otimes_R F') & \xleftarrow{\cong} & \text{Tor}_i^R(L, M) \\ \downarrow H(L \otimes_R t_j^{F'}) & & \downarrow \epsilon_j^{\text{right}} \\ H_{i-2}(L \otimes_R F') & \xleftarrow{\cong} & \text{Tor}_{i-2}^R(L, M) \end{array}$$

where $F' \xrightarrow{\cong} M$ is a free resolution of the R -module M , $\vartheta: F \rightarrow F'$ is a comparison of resolutions, and the isomorphisms are $H\text{Hom}_R(\vartheta, \eta_R^N)$ and $H(\varepsilon_L^R \otimes_R \vartheta)$. Operators ϵ_j^{left} are similarly defined on $\text{Tor}_i^R(L, M)$, by means of a free resolution $E' \xrightarrow{\cong} L$.

PROPOSITION. *On $\text{Ext}_R^*(M, N)$ there are equalities $\chi_j = \epsilon_j^{\text{right}}$ for $j = 1, \dots, c$.*

On $\text{Tor}_^R(L, M)$ there are equalities $\chi_j = \epsilon_j^{\text{left}} = -\epsilon_j^{\text{right}}$ for $j = 1, \dots, c$.*

Remark. This corrects a sign error in [7, Proposition 1.6], which asserts that $\epsilon_j^{\text{left}} = \epsilon_j^{\text{right}}$. The argument given there may not be used to compare the two maps: it assumes incorrectly that the canonical projections

$$E' \otimes_R M \leftarrow E' \otimes_R F' \rightarrow L \otimes_R F'$$

commute with the chain endomorphisms

$$t_j^{E'} \otimes_R M, t_j^{E'} \otimes_R F' + E' \otimes_R t_j^{F'}, E' \otimes t_j^{F'}.$$

Proof. For an R -free resolution $F \xrightarrow{\cong} M$, pick a homogenous basis $\{f'_\lambda\}_{\lambda \in \Lambda}$ of the graded R -module F'^h , and let P^h be a graded K^h -module on a basis $\{g_\lambda \mid |g_\lambda| = |f'_\lambda|\}_{\lambda \in \Lambda}$. Set $\omega_{<0} = 0$, and assume by induction that a κ -covariant morphism $\omega_{<h}: P_{<h} = \coprod_{|g_\lambda| < h} K^h g_\lambda \rightarrow F'_{<h}$ has been constructed such that $\omega_{<h}(g_\lambda) = f'_\lambda$ when $|g_\lambda| < h$, and $\omega_{<h}$ is a surjective quasi-isomorphism.

If $|f'_\lambda| = h$, then $\partial(f'_\lambda)$ is a cycle in $\text{Im}(\omega_{<h})$, hence $\partial(f'_\lambda) = \omega_{<h}(z_\lambda)$ for some cycle $z_\lambda \in P_{<h}$. Extend the differential from $P_{<h}$ to $P_{\leq h}$ by setting $\partial(g_\lambda) = z_\lambda$, and extend $\omega_{<h}$ to $\omega_{\leq h}: P_{\leq h} \rightarrow F'_{\leq h}$ by $\omega_{\leq h}(g_\lambda) = f'_\lambda$ for all λ with $|g_\lambda| \leq h$. By the choices made $\omega_{\leq h}$ is a morphism and by the Five-Lemma it is a quasi-homomorphism onto its image. The induction step is complete. Now set $P = \varinjlim_h P_{<h}$ and $\omega = \varinjlim_h \omega_{<h}: P \rightarrow F'$.

Clearly, ω is a κ -covariant surjective morphism, and the induced map $R \otimes_K P \rightarrow F'$ is an isomorphism of complexes of R -modules. Using the notation of Subsection 2.4, we see that the pair $(P^h/K_+^h P^h, \partial_\varnothing)$ is a lifting of the complex F' . In view of (2.4.1) we can choose $\partial_1, \dots, \partial_c$ to be the chain endomorphisms associated with this lifting. This choice leads to equalities of Eisenbud's endomorphism with those from Subsection 2.4:

$$(4.2.1) \quad t_j^{F'} = \tau_j^{F'} \quad \text{for } j = 1, \dots, c.$$

Lemma 2.5 with $\varphi = \text{id}_R$ and $\gamma = \vartheta$ shows that $\vartheta\tau_j^F$ and $\tau_j^{F\vartheta}$ are homotopic, hence

$$\text{H Hom}_R(\vartheta, \eta_R^N) \circ \text{H Hom}_R(t_j^{F'}, N) = \text{H Hom}_R(\tau_j^F, J) \circ \text{H Hom}_R(\vartheta, \eta_R^N).$$

As $\text{H Hom}_R(\tau_j^F, J) = \chi_j$, we see that $\chi_j = \epsilon_j^{\text{right}}$. The argument for Tor is similar. ■

4.3. *Avramov's Construction* [1, Sect. 1]. This dualizes that of Eisenbud described in Subsection 4.2.

Recall that J is the chosen injective resolution of the R -module N . Choose a graded Q -module \tilde{J} such that the Q -module \tilde{J}_i is an injective envelope of the Q -module J_i for each i . Clearly, $\text{Hom}_Q(R, \tilde{J}) \cong J$. Since \tilde{J} is a graded injective Q -module, there exists an endomorphism $\tilde{\partial}$ of degree -1 on \tilde{J} such that $\tilde{\partial}|_J = \partial$. By [1, Proposition 1.2] there are degree -2 endomorphisms $\tilde{u}_j(\tilde{J}, \mathbf{x}): \tilde{J} \rightarrow \tilde{J}$, which satisfy $\tilde{\partial}^2 = \sum_{j=1}^c x_j \tilde{u}_j(\tilde{J}, \mathbf{x})$, and which restrict to degree -2 chain endomorphisms $u_j^J: J \rightarrow J$.

The operators $\epsilon_j^{\text{right}}$ studied in [1] are defined by commutativity of the square

$$\begin{array}{ccc} \text{Ext}_R^i(M, N) & \xleftarrow{\cong} & \text{H}_{-i} \text{Hom}_R(M, J) \\ \epsilon_j^{\text{right}} \downarrow & & \downarrow \text{H Hom}_R(M, u_j^J) \\ \text{Ext}_R^{i+2}(M, N) & \xleftarrow{\cong} & \text{H}_{-i-2} \text{Hom}_R(M, J) \end{array}$$

with isomorphisms $\text{H Hom}_R(\epsilon_M^R, J)$.

PROPOSITION. On $\text{Ext}_R^*(M, N)$ there are equalities $\chi_j = \epsilon_j^{\text{right}}$ for $j = 1, \dots, c$.

Remark. Comparison with Proposition 4.2 shows that $\epsilon_j^{\text{left}} = \epsilon_j^{\text{right}}$. This is stated in [1, (1.5)], as a consequence of [1, Proposition 1.4]. However, that proposition is not correct: the morphisms $\text{Hom}_R(F, \eta_R^N)$ and $\text{Hom}_R(\epsilon_M^R, J)$ do not commute with the actions induced by the chain endomorphisms t_j^F and u_j^J .

Proof. With the notation of Subsection 2.5, we set $\underline{Y} = (0: K_+^{\natural})_Y$ and consider the endomorphisms $\partial_H: \underline{Y} \rightarrow \underline{Y}$ constructed in Subsection 2.5. Because of the equalities $J = (0: \alpha)_{\underline{Y}} = (0: \mathfrak{A})_{\underline{Y}}$ and $\partial_J = \partial_{\mathfrak{A}}|_J = \partial_Y|_J$, there is an injective homomorphism of graded Q -modules $\tilde{\iota}: \tilde{J} \rightarrow \underline{Y}$ with $\tilde{\iota}|_J = \text{id}_J$ and $\tilde{\iota} \circ \tilde{\partial} - \partial_{\mathfrak{A}} \circ \tilde{\iota} = \sum_{j=1}^c x_j h_j$ for appropriate degree -1 homomorphisms of graded Q -modules $h_j: \tilde{H} \rightarrow \underline{Y}$. A direct computation gives

$$\sum_{j=1}^c x_j (\tilde{\iota} \circ \tilde{u}_j^J(J, \mathbf{x}) - v_j^J \circ \tilde{\iota}) = \sum_{j=1}^c x_j (h_j \circ \tilde{\partial} + \partial_{\mathfrak{A}} \circ h_j).$$

As the Koszul complex is exact, u_j^J is homotopic to v_j^J , yielding the middle equality

$$\epsilon_j^{\text{right}} = \text{H Hom}_R(F, u_j^J) = \text{H Hom}_R(F, v_j^J) = \chi_j;$$

the last one comes from Proposition 2.5 ■

4.4. *Mehta's Construction* [9, Sect. 1]. This proceeds from a presentation

$$0 \rightarrow B \rightarrow G_0 \xrightarrow{\varepsilon} M \rightarrow 0$$

with a projective Q -module G_0 .

In view of the isomorphism $\text{Tor}_1^Q(R, G_0) = 0$ and $\text{Tor}_1^Q(R, M) \cong C \otimes M$, where $C = \alpha/\alpha^2$ is the cotangent module, it induces an exact sequence of R -modules

$$\mathcal{E}^M \quad 0 \rightarrow C \otimes M \xrightarrow{\iota} B' \rightarrow G'_0 \xrightarrow{\varepsilon'} M \rightarrow 0,$$

where $-' = R \otimes -$, and $C = \alpha/\alpha^2$ is the cotangent module, cf. Subsection 2.1. In the notation introduced there, the homomorphism $\iota(\sum_k \bar{x}_k \otimes m_k) = 1 \otimes \sum_k x_k m'_k$, where $m'_k \in G_0$ satisfy $\varepsilon(m'_k) = m_k$. For each j we get an exact sequence

$$\mathcal{E}_j^M \quad 0 \rightarrow M \rightarrow M_j \rightarrow G'_0 \xrightarrow{\varepsilon'} M \rightarrow 0,$$

as the pushout of \mathcal{E}^M along $x_j^* \otimes M: C \otimes_R M \rightarrow R \otimes_R M = M$, where x_1^*, \dots, x_c^* is the basis of C^* dual to $\bar{x}_1, \dots, \bar{x}_c$.

The iterated connecting homomorphisms induced by \mathcal{E}_j^M yield operators δ_j^{right} on $\text{Ext}_R^*(M, N)$ and δ_j^{left} on $\text{Tor}_*^R(L, M)$ studied by Mehta. Furthermore, the exact sequences \mathcal{E}_j^N and \mathcal{E}_j^L yield operators δ_j^{left} on $\text{Ext}_R^*(M, N)$ and δ_j^{right} on $\text{Tor}_*^R(L, M)$.

PROPOSITION. On $\text{Ext}_R^*(M, N)$ there are equalities $\chi_j = \delta_j^{\text{left}} = -\delta_j^{\text{right}}$ for $1 \leq j \leq c$.

On $\text{Tor}_*^R(L, M)$ there are equalities $\chi_j = -\delta_j^{\text{left}} = \delta_j^{\text{right}}$ for $1 \leq j \leq c$.

Remarks. (1) The last statement corrects a sign error in [9, Proposition 2.4], which asserts that $\delta_j^{\text{eft}} = \delta_j^{\text{right}}$ on $\text{Tor}_*^R(L, M)$.

(2) Mehta also considers an analogous exact sequence obtained as the pullback of an injective corepresentation of M ; when $c = 1$ he proves that these sequences are congruent with \mathcal{E}_j^M , and hence the same cohomology operators. His argument in [9, Proposition 2.5] can be extended to handle the case of arbitrary codimension.

Proof. Let (G, ∂) be a free resolution of M over Q . As $x_j M = 0$, for each j there is a homotopy σ^j from $x_j \text{id}_G$ to 0_G . In the diagram of

R-modules

$$\begin{array}{ccccccc}
(C \otimes G_1) \oplus G'_3 & \xrightarrow{\begin{bmatrix} C \otimes \partial_1 & 0 \\ \sigma_1 & \partial'_3 \end{bmatrix}} & (C \otimes G_0) \oplus G'_2 & \xrightarrow{[\sigma_0 \quad \partial'_2]} & G'_1 & \xrightarrow{\partial'_1} & G'_0 \xrightarrow{\varepsilon'} M \rightarrow 0 \\
\downarrow & & \downarrow [C \otimes \varepsilon' \quad 0] & & \downarrow \pi & & \parallel \\
0 & \xrightarrow{\quad} & C \otimes M & \xrightarrow{\iota} & B' & \rightarrow & G'_0 \xrightarrow{\varepsilon'} M \rightarrow 0 \\
& & \downarrow x_j^* \otimes M & & \downarrow & & \parallel \\
0 & \xrightarrow{\quad} & M & \xrightarrow{\quad} & M_j & \rightarrow & G'_0 \xrightarrow{\varepsilon'} M \rightarrow 0
\end{array}$$

the middle and lower rows are the exact sequences \mathcal{E}^M and \mathcal{E}_j^M above, π comes from the canonical factorization of ∂'_1 and $\sigma_i(\sum_k \bar{x}_k \otimes g_k) = 1 \otimes \sum_k \sigma_i^k(g_k)$ for $i = 0, 1$.

The commutativity of the upper left hand square is clear. The one to its right commutes because $\partial_1 \sigma_0^j = x_j \text{id}_G$. Thus, the diagram is commutative. It is checked directly that the first row is exact.² We extend it to a free resolution F' of M over R , and identify $\text{Ext}_R^*(M, M)$ with $\text{H Hom}_R(F', I)$ via the canonical isomorphism.

Let $\mu_j^M: F' \rightarrow I$ be the chain map which on F'_2 is the composition of $x_j^* \otimes \bar{\varepsilon}$ with the embedding $\eta: M \rightarrow I_0$, and is trivial elsewhere. The commutativity of the diagram implies that $-\mu_j^M$ is the extension class associated with the exact sequence \mathcal{E}_j^M , cf. [6, Sect. 7, No. 3, Definition 1]. Now Bourbaki [6, Sect. 7, No. 6, Corollary 3] yields

$$\delta_j^{\text{left}}(\gamma) = -\gamma \cdot [-\mu_j^M] \quad \text{and} \quad \delta_j^{\text{right}}(\gamma) = [-\mu_j^N] \cdot \gamma$$

for $\gamma \in \text{Ext}_R^*(M, N)$,

while [6, Sect. 7, No. 8, Corollaries 2 and 4] show that

$$\delta_j^{\text{right}}(t) = [-\mu_j^M] \cdot t \quad \text{and} \quad \delta_j^{\text{left}}(t) = [-\mu_j^L] \cdot t$$

for $t \in \text{Tor}_*^R(L, M)$.

On the other hand, lifting $F'_2 \rightarrow F'_1 \rightarrow F'_0$ to homomorphisms of free Q -modules

$$G_0^c \oplus G_2 \xrightarrow{\tilde{\partial}_2} G_1 \xrightarrow{\tilde{\partial}_1} G_0$$

with $\tilde{\partial}_1 = \partial_1$ and $\tilde{\partial}_2((g_1, \dots, g_c) + g) = \sum_k \sigma_0^k(g_k) + \partial^2(g)$, we see that Eisenbud's operator $t_j^{F'}$ from Subsection 4.2 acts on F'_2 by the formula $t_j^{F'}(\sum_k \bar{x}_k \otimes g_k + 1 \otimes g) = 1 \otimes g_j$. This implies $[\mu_j^M] = [\eta_R^M \circ \varepsilon_M^R \circ t_j^{F'}]$. As $t_j^{F'} = \tau_j^{F'}$ by (3.2.1), Corollary 2.4 and Theorem 2.7 yield $\chi_j = \delta_j^{\text{left}} = -\delta_j^{\text{right}}$ on $\text{Ext}_R^*(M, N)$ and $\chi_j = \delta_j^{\text{right}} = -\delta_j^{\text{left}}$ on $\text{Tor}_*^R(L, M)$. ■

² In fact, it represents the beginning of a free resolution of M over R , constructed from G by Eisenbud [7], but we shall not need the rest of it.

5. APPLICATION

We extend the important finiteness result discussed in the Introduction and its recent converse to Ext 's of complexes of R -modules. We also establish a new property of the cohomology operators—their primitivity.

For the first two subsections we keep the notation and hypothesis of Subsection 2.1, and as in Proposition 3.3 denote by \mathcal{S} the symmetric algebra of the R -module $\text{Hom}_R(\alpha/\alpha^2, R)$, graded by assigning to its generators cohomological degree 2.

5.1. Finiteness. The R -module $\text{Ext}_Q^*(M, N)$ is noetherian when $\text{Ext}_Q^n(M, N)$ is noetherian over R (or, equivalently, over Q) for each n , and vanishes for $n \gg 0$.

THEOREM. *If M and N are complexes of R -modules, then the R -module $\text{Ext}_Q^*(M, N)$ is noetherian if and only if the \mathcal{S} -module $\text{Ext}_R^*(M, N)$ is noetherian.*

Comments in Place of Proof. Eisenbud's constructions of cohomology operators ϵ_j^{left} , recalled in Subsection 4.2, can easily be extended to complexes, and the proof that they coincide with the cohomology operators χ_j carries over. Thus, by Theorem 3.3, we may consider $\text{Ext}_R^*(M, N)$ as a module over the ring \mathcal{S} of Eisenbud operators.

If $\text{Ext}_Q^*(M, N)$ is noetherian over Q , then the spectral sequence argument given in [2] for finiteness over \mathcal{S} shows that $\text{Ext}_R^*(M, N)$ is noetherian over \mathcal{S} . (If $H(M)$ is bounded below and $H(N)$ is bounded above, then Gulliksen's original proof of [8, Theorem 3.1] can also be applied, in view of Proposition 3.1.)

If M and N are Q -modules and $\text{Ext}_R^*(M, N)$ is noetherian over \mathcal{R} , then it is proved in [4, Theorem 4.2] that $\text{Ext}_Q^*(M, N)$ is noetherian over Q . That argument only depends on the formal properties of the spectral sequence of [4, Theorem 4.4], whose construction works equally well for complexes. ■

Let $\mathcal{Z}_R^*(M)$ denote the subalgebra of $\text{Ext}_R^*(M, M)$ generated over $R/\text{ann}(M) \subseteq \text{Hom}_R(M, M)$ by the central elements of degree 2. From Theorems 4.1 and 2.7 we get

COROLLARY. *If the Q -module $\text{Ext}_Q^*(M, N)$ is noetherian, then the $\mathcal{Z}_R^*(M)$ - $\mathcal{Z}_R^*(N)$ -bimodule $\text{Ext}_R^*(M, N)$ is noetherian over each ring.*

5.2. Primitivity. A DG algebra with divided powers (or DG Γ -algebra) over R is a graded commutative DG R -algebra A with $A_i = 0$ for $i \leq 0$, equipped with maps $A_{2n} \ni a \mapsto a^{(i)} \in A_{2ni}$, defined for all $n \geq 1$ and $i \geq 0$, and satisfying some standard conditions, cf. e.g., [5, (1.3)]. A Γ -derivation is an R -linear map $\theta: A \rightarrow A$ such that

$$\theta(a'a'') = \theta(a')a'' + (-1)^{|\theta||a'|} a'\theta(a'') \quad \text{and} \quad \theta(a^{(i)}) = \theta(a)a^{(i-1)}.$$

Let S and T be DG Γ algebras over R . An extension of Tate's procedure [10] of adjunction of Γ -variables to R factors the structure homomorphism $R \rightarrow S$ as the composition of $R \rightarrow E$, where $E^\sharp = R\langle X \rangle$ is a free Γ -algebra over R , with a surjective quasi-isomorphism $E \xrightarrow{\sim} S$. By [5, (2.16)], $\text{Tor}_*^R(S, T) \simeq H_*(E \otimes_R T)$ inherits a structure of Γ -algebra, which is independent of the choice of E .

THEOREM. On $\text{Tor}_*^R(S, T)$ the operator χ_j is a Γ -derivation for $1 \leq j \leq c$.

Proof. Consider a graded Γ -algebra $\tilde{E} = Q\langle \tilde{X} \rangle$, with \tilde{X} a new set of divided power variables in bijective, in degree preserving correspondence $\tilde{x} \leftrightarrow x$ with those of X . This correspondence extends to an isomorphism of Γ -algebras $\tilde{E} \otimes_Q R = E^\sharp$. For each $x \in X$, choose in \tilde{E} an element x' such that $x' \otimes 1 = \partial(x)$, and let $\tilde{\partial}$ be the unique Q -linear Γ -derivation of \tilde{E} with $\tilde{\partial}(\tilde{x}) = x'$. It is easy to see that $\tilde{\partial}^2$ is a Γ -derivation.

For each $\tilde{x} \in \tilde{X}$ there are $e_j(\tilde{x}) \in \tilde{E}$ such that $\tilde{\partial}^2(\tilde{x}) = \sum_j x_j e_j(\tilde{x})$. Let \tilde{t}_j be the unique Q -linear Γ -derivation of \tilde{E} such that $\tilde{t}_j(\tilde{x}) = e_j(\tilde{x})$ for all $\tilde{x} \in \tilde{X}$. As $\tilde{\partial}^2$ and $\sum_{j=1}^c x_j \tilde{t}_j$ agree on the Γ -generators of the Q -algebra \tilde{E} , these two endomorphisms of \tilde{E} coincide. Thus, $t_j^E = \tilde{t}_j \otimes_Q R$ is a Γ -derivation of $\tilde{E} \otimes_Q R = E$, and hence $H(t_j^E \otimes_R T)$ is a Γ -derivation of $\text{Tor}_*^R(S, T)$.

As $\chi_j = H(t_j^E \otimes_R T)$ by Proposition 3.2, this proves our assertion. ■

We say that $\gamma \in \text{Ext}_R^n(S, S)$ is *primitive* if $\gamma(tu) = 0$ whenever $t \in \text{Tor}_i^R(S, S)$ and $u \in \text{Tor}_{n-i}^R(S, S)$ with $0 < i < n$, and $\gamma(t^{(i)}) = 0$ for all $t \in \text{Tor}_{2n}^R(S, S)$ with $n > 0$ and $i \geq 2$. Primitives form a graded R -submodule $\text{Pr}_R^*(S) \subseteq \text{Ext}_R^*(S, S)$, and $\text{Pr}_R^2(S) = \text{Ker}(\pi_R^S)$, where π_R^S is the homomorphism of R -modules

$$\pi_R^S: \text{Ext}_R^2(S, S) \rightarrow \text{Hom}_R(\text{Tor}_1^R(S, S)^2, \text{Tor}_0^R(S, S))$$

$$\text{with } \pi_R^S(\gamma)(g) = \gamma \cdot t.$$

We denote $\mathcal{P}_R^*(S)$ the subalgebra of $\text{Ext}_R^*(S, S)$ generated over $R/\text{ann}(S)$ by the *central primitive elements of degree 2*. The last theorem and Corollary 5.1 yield

COROLLARY. If T is a DG Γ -algebra over R and M is a complex of R -modules such that $\text{Ext}_Q^*(M, T)$ is noetherian over Q , then $\text{Ext}_R^*(M, T)$ is noetherian over $\mathcal{P}_R^*(T)$.

5.3. CI-Dimension. Let R be a noetherian local ring with residue field k .

A *quasi-deformation* of R is a diagram of homomorphisms $R \rightarrow R' \leftarrow Q$ of local rings, with $R \rightarrow R'$ faithfully flat and $R' \leftarrow Q$ surjective with kernel generated by a regular sequence. An R -module $M \neq 0$ is said to have *finite CI-dimension*, cf. [4], if R has a quasi-deformation such that $M \otimes_R R'$ has finite projective dimension over Q .

THEOREM. *If R is a local ring with residue field k and M is a finite R -module of finite CI-dimension, then the graded left module $\text{Ext}_R^*(M, k)$ is finite over the k -subalgebra $\mathcal{P}_R^*(k) \subseteq \text{Ext}_R^*(k, k)$ generated by the central primitive elements of degree 2.*

Proof. We set $S = k \otimes_R R'$ and make the identification $\text{Ext}_R^*(k, k) \otimes_k S = \text{Ext}_{R'}^*(S, S)$ through the canonical isomorphism due to the flatness of R' over R . We have

$$\mathcal{Z}_R^*(k) \otimes_k S = \mathcal{Z}_{R'}^*(S)$$

since the center of a tensor product of algebras over a field is equal to the tensor product of the centers of the factors. The flatness of R' also identifies $\text{Tor}_*^R(k, k) \otimes_k S$ and $\text{Tor}_*^{R'}(S, S)$ in a way compatible with products in Tor and the action of Ext, so

$$\text{Pr}_R(k) \otimes_R S = \text{Ker}(\pi_R^k) \otimes_R S = \text{Ker}(\pi_{R'}^S) = \text{Pr}_{R'}(S)$$

and we conclude that

$$\mathcal{P}_R^*(k) \otimes_k S = \mathcal{P}_{R'}^*(S).$$

Identifying $\text{Ext}_R^*(M, k) \otimes_k S$ with $\text{Ext}_{R'}^*(M \otimes_R R', S)$, and noting that by Corollary 5.2 the latter is a noetherian module over $\mathcal{P}_{R'}^*(S)$. We conclude by flat descent that $\text{Ext}_R^*(M, k)$ is noetherian over $\mathcal{P}_R^*(k)$. ■

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